

# Analytic Description of the Motion of a Trapped Ion in an Even or Odd Squeezed State

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## ABSTRACT

A completely analytic description is given of the motion of a trapped ion which is in either an even or an odd squeezed state. Comparison is made to recent results on the even or odd coherent states, and possible experimental work is discussed.

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# 1 Introduction

The even (+) and odd (−) coherent states [1] can be defined as the eigenstates of the double-destruction operator,  $aa$ :

$$aa|\alpha\rangle_{\pm} = \alpha^2|\alpha\rangle_{\pm} . \quad (1)$$

They explicitly are

$$|\alpha\rangle_+ = [\cosh |\alpha|^2]^{-1/2} \sum_{n=0}^{\infty} \frac{\alpha^{2n}}{\sqrt{(2n)!}} |2n\rangle \rightarrow \psi_+ , \quad (2)$$

$$|\alpha\rangle_- = [\sinh |\alpha|^2]^{-1/2} \sum_{n=0}^{\infty} \frac{\alpha^{2n+1}}{\sqrt{(2n+1)!}} |2n+1\rangle \rightarrow \psi_- , \quad (3)$$

where we will go back and forth between Dirac and wave-function notation. These states also are the appropriate minimum-uncertainty coherent states.

They can also be created by a special displacement operator [1, 2]:

$$|\alpha\rangle_{\pm} = D_{\pm}(\alpha)|0\rangle = \left[2(1 \pm \exp[-2|\alpha|^2])\right]^{-1/2} [D(\alpha) \pm D(-\alpha)]|0\rangle . \quad (4)$$

where  $D$  is the ordinary coherent state displacement operator:

$$D(\alpha) = \exp[\alpha a^{\dagger} - \alpha^* a] , \quad \alpha = \alpha_1 + i\alpha_2 \equiv (x_0 + ip_0)/\sqrt{2} . \quad (5)$$

Matos Filho and Vogel [3] have recently given a dynamical analysis, as a function of time, of a trapped ion which, to very high precision, is in an even or odd coherent state. (Such a system has been produced experimentally by Wineland's group [4].) They gave lovely three-dimensional numerical graphs of the probability densities and Wigner functions, for the even and odd cases, as functions of position and time, for particular values of  $\alpha$ .

Previously, we had observed that closed-form expressions can be given for these wave functions in the time-independent case [5, 6]:

$$\psi_+ = \left[ \frac{e^{-\alpha^2}}{\pi^{1/2} \cosh |\alpha|^2} \right]^{1/2} e^{-x^2/2} \cosh(\sqrt{2}\alpha x) , \quad (6)$$

$$\psi_- = \left[ \frac{e^{-\alpha^2}}{\pi^{1/2} \sinh |\alpha|^2} \right]^{1/2} e^{-x^2/2} \sinh(\sqrt{2}\alpha x) . \quad (7)$$

These expressions can be put in the form of two Gaussians displaced on opposite sides of the origin:

$$\psi_{\pm} = \left[ 2\pi^{1/2} (1 \pm e^{-2|\alpha|^2}) \right]^{-1/2} \left[ e^{-(x-\sqrt{2}\alpha_1)^2/2+i\sqrt{2}\alpha_2 x} \pm e^{-(x+\sqrt{2}\alpha_1)^2/2-i\sqrt{2}\alpha_2 x} \right] . \quad (8)$$

where we have ignored  $e^{-i4\alpha_1\alpha_2}$ . This is an intuitively satisfying representation.

Then noting that time displacement can be included by letting  $\alpha \rightarrow \alpha \exp[-i\omega t]$ , we then could obtain an analytic expression for the wave functions as a function of time [7]. Taking the convention  $\alpha \rightarrow \alpha_0$  is real, as was done in Ref. [3], the probability densities as a function of time were shown to be

$$\rho_+ = \frac{e^{\alpha_0^2[\sin^2 \omega t - \cos^2 \omega t]}}{\pi^{1/2}[e^{\alpha_0^2} + e^{-\alpha_0^2}]} e^{-x^2} [\cosh\{2\sqrt{2}\alpha_0(\cos \omega t)x\} + \cos\{2\sqrt{2}\alpha_0(\sin \omega t)x\}], \quad (9)$$

$$\rho_- = \frac{e^{\alpha_0^2[\sin^2 \omega t - \cos^2 \omega t]}}{\pi^{1/2}[e^{\alpha_0^2} - e^{-\alpha_0^2}]} e^{-x^2} [\cosh\{2\sqrt{2}\alpha_0(\cos \omega t)x\} - \cos\{2\sqrt{2}\alpha_0(\sin \omega t)x\}]. \quad (10)$$

The Wigner functions can be obtained similarly.

The above  $\rho_+$  and  $\rho_-$  described the forms of Figs. 1 and 4 in Ref. [3] and we show them in our Figs 1 and 2 (in time units of  $\omega$ , i.e.  $\omega = 1$ ). The terms  $\exp[-x^2] \times \cosh$  describe the two “wave-packets” on opposite sides of the origin. Until they intersect, these wave-packets resemble the non-spreading evolution of ordinary coherent states. The  $\cos$  terms describe the interference effects near  $x = 0$  at  $t = (2j+1)\pi/2$ . The even and odd natures are manifested by the maximum or zero at the origin, respectively, and the symmetry of the humps about the origin. (Other discussions of “Schrödinger Cat” or “two-packet” states should also be consulted [8].)

The question now arises if this formalism can be extended to squeezed states.

## 2 Squeezed States

For the even and odd systems, there is a well-defined mathematical prescription to obtain ladder-operator and equivalent minimum-uncertainty squeezed states [5]. They

are given, explicitly, as the eigenstates of the equation

$$\left[ \left( \frac{1+q}{2} \right) aa + \left( \frac{1-q}{2} \right) a^\dagger a^\dagger \right] \psi_{ss} = \alpha^2 \psi_{ss}. \quad (11)$$

The solutions are [5]

$$\psi_{Ess} = N_E \exp \left[ -\frac{x^2}{2}(q + \sqrt{q^2 - 1}) \right] \Phi \left( \left[ \frac{1}{4} + \frac{\alpha^2}{2\sqrt{q^2 - 1}} \right], \frac{1}{2}; x^2 \sqrt{q^2 - 1} \right), \quad (12)$$

$$\psi_{Oss} = N_O x \exp \left[ \frac{-x^2}{2}(q + \sqrt{q^2 - 1}) \right] \Phi \left( \left[ \frac{3}{4} + \frac{\alpha^2}{2\sqrt{q^2 - 1}} \right], \frac{3}{2}; x^2 \sqrt{q^2 - 1} \right), \quad (13)$$

where  $\Phi(a, b; c)$  is the confluent hypergeometric function  $\sum_{n=0}^{\infty} \frac{(a)_n c^n}{(b)_n n!}$ . In the limit  $q \rightarrow 1$ , these become the even and odd coherent states.

For the ordinary harmonic-oscillator coherent and squeezed states, there are equivalent displacement-operator squeezed states, since there exists a unitary Bogoliubov-type squeeze operator:

$$S(z) = \exp \left[ \frac{1}{2} z a^\dagger a^\dagger - \frac{1}{2} z^* a a \right], \quad z = r e^{i\phi} = z_1 + i z_2, \quad (14)$$

with the property

$$S^\dagger a S = (\cosh r) a + e^{i\phi} (\sinh r) a^\dagger. \quad (15)$$

In wave-function form, these states are [9]

$$D(\alpha) S(z) |0\rangle = \frac{\exp[-\frac{i}{2} x_0 p_0]}{\pi^{1/4} [s(1 + i2\kappa)]^{1/2}} \exp \left[ -(x - x_0)^2 \left( \frac{1}{2s^2(1 + 12\kappa)} - i\kappa \right) + i p_0 x \right], \quad (16)$$

$$s \equiv \cosh r + \frac{z_1}{r} \sinh r, \quad \kappa \equiv \frac{z_2 \sinh r}{2rs}. \quad (17)$$

(For  $z$  real,  $\ln s = r \operatorname{sgn}(r)$ .)

But there are no equivalent displacement-operator squeezed states for the even/odd systems, because there is no unitary operator that can transform  $aa$  into the operator of Eq. (11). However, an alternate idea is to simply use  $S$  for the coherent even/odd systems,

$$\psi_{s\pm} = D_\pm(\alpha) S(z) |0\rangle \quad (18)$$

on the physical grounds that  $S$  is the dilation operator [10]. If one does that, then each of the packets of the even/odd states will be of the form of Eq. (16). Taking, for simplicity, the case  $z$  real (or  $\kappa = 0$ ), these states are (ignoring an overall phase)

$$\psi_{s\pm} = \left[ \pi^{1/2} 2s (1 \pm e^{-x_0^2/s^2 - p_0^2 s^2}) \right]^{-1/2} \left[ e^{-(x-x_0)^2/(2s^2) + ip_0 x} \pm e^{-(x+x_0)^2/(2s^2) - ip_0 x} \right]. \quad (19)$$

Now we compare  $\psi_{E/Oss}$  with  $\psi_{s\pm}$ . We do this in Fig. 3. There we compare  $\rho_{Ess} = \psi_{Ess}^* \psi_{Ess}$ , having parameters  $\alpha = 2$  and  $q = 2$ , with  $\rho_{s+} = \psi_{s+}^* \psi_{s+}$ , having parameters  $\alpha = 2$  (or  $x_0 = 2\sqrt{2}$ ,  $p_0 = 0$ ) and  $s = 3/2$ . These parameters were chosen not for the best overlap, but for a simple comparison of shapes. One sees that the two probability densities are quite similar.  $\rho_{Oss}$  is even more similar to  $\rho_{s-}$  because both are constrained to go to zero at the origin.

Therefore, because of this similarity, and the analytic exactness of the  $\psi_{s\pm}$  system, we now proceed with this choice of squeezed states.

### 3 Time Evolution

For the squeezed even/odd system, one no longer has the simple criterion that  $\alpha \rightarrow \alpha e^{-i\omega t}$  describes the wave function.

Instead we choose to consider the unitary time-evolution operator (time again in units of  $\omega$ )

$$T = \exp[-iHt] = \exp[-i(a^\dagger a + 1/2)t] = \exp[-i(x^2 - \partial^2)/2]. \quad (20)$$

To make this operator useful, one can transform it to coordinates of the second kind with Baker-Campbell-Hausdorff relations. (BCH relations are usually obtained in terms of raising and lowering operators, not in terms of the functional operators we have here.) But when this is done, one obtains [9]

$$T = [\cos t]^{-1/2} \exp\left[-\frac{i}{2} \tan t (x^2)\right] \exp[-(\ln \cos t)(x\partial)] \exp\left[\frac{i}{2} \tan t (\partial^2)\right], \quad (21)$$

where the operational definitions on a function  $h(x)$  are

$$\exp[\tau(x\partial)]h(x) = h(xe^\tau) \quad (22)$$

$$\exp[c(\partial^2)]h(x) = \frac{1}{[4\pi c]^{1/2}} \int_{-\infty}^{\infty} \exp\left[-\frac{(y-x)^2}{4c}\right] h(y) dy . \quad (23)$$

With this result, one can calculate

$$\psi_{s\pm}(t) = U\psi_{s\pm} . \quad (24)$$

Taking, for simplicity, the case  $z$  is real (or  $\kappa = 0$ ) one has

$$\begin{aligned} \psi_{s\pm}(t) = & \left[ \frac{s}{2\pi^{1/2}(1 \pm e^{-x_0^2 \cos^2 t})} \frac{s^2 \cos t - i \sin t}{s^4 \cos^2 t + \sin^2 t} \right]^{1/2} \\ & \left\{ \exp \left[ -\frac{(x - x_0 \cos t)^2}{2} \left( \frac{s^2 - i \tan t}{s^4 \cos^2 t + \sin^2 t} \right) - \frac{i}{2}(\tan t)x^2 \right] \right. \\ & \left. \pm \exp \left[ -\frac{(x + x_0 \cos t)^2}{2} \left( \frac{s^2 - i \tan t}{s^4 \cos^2 t + \sin^2 t} \right) - \frac{i}{2}(\tan t)x^2 \right] \right\} . \quad (25) \end{aligned}$$

The terms  $\exp[-i(\tan t)x^2/2]$  turn out to be necessary to cancel the singularities of the terms  $\exp[ix^2 \tan t/(2 \sin^2 t)]$  when  $t$  is an odd multiple of  $\pi/2$ .

Then some algebra yields

$$\begin{aligned} \rho_{s\pm} = & \frac{\exp[-(x^2 + x_0^2 \cos^2 t)/d^2]}{\pi^{1/2} d [1 \pm d \exp[-x_0^2/s^2]]} \\ & \left\{ \cosh \left( \frac{2xx_0(\cos t)}{d^2} \right) \pm \cos \left( \frac{2xx_0 \sin t}{d^2 s^2} \right) \right\} , \quad (26) \end{aligned}$$

where

$$d^2 = s^2 \cos^2 t + \sin^2 t / s^2 . \quad (27)$$

## 4 Discussion

In Figures 4 and 5 we plot the probability densities  $\rho_{s+}$  and  $\rho_{s-}$ , respectively, as functions of  $x$  and  $t$ . This is done for parameters  $x_0 = 4$  and  $s = 2$ . This value of  $s$  means the wave packets have a large  $x$  uncertainty at  $t = 0$ , when they are separated.

However, when they collide at the origin at  $t = \pi/2$ , their widths are narrow, and so the interference peak is much larger and much more confined than was the case with the coherent states ( $s = 1$ ). Figure 6 has the same type of description, except that since this is an odd state, there is a null at the origin when  $t = \pi/2$ , so that there are two smaller narrow peaks about the origin, which however are still much taller than the coherent-state peaks.

In Figures 6 and 7 we plot the probability densities  $\rho_{s+}$  and  $\rho_{s-}$ , respectively, as functions of  $x$  and  $t$ . This time it is done for parameters  $x_0 = 4$  but  $s = 1/2$ . This value of  $s$  means the separated wave packets at  $t = 0$  have a small  $x$  uncertainty. Therefore, when the packets collide at the origin at  $t = \pi/2$ , they have a large  $x$  uncertainty, and so the interference pattern is very pronounced and broad. The evenness and oddness of the two figures is reflected by their being a single hump at the origin in Figure 6 and a null, surrounded symmetrically by humps, in Figure 7.

The cases  $s = 2, 1/2$ , are three examples in a continuum for  $|z| = 2$ . the first case has the phase  $\phi = 0$  and the second case has  $\phi = \pi$ . All other  $\phi$  represent cases where the squeezing is rotated between the  $s$  and  $p$  phase-space coordinates, and so the maximum heights of the wave packets occur at times different than  $t = 0$  or  $\pi/2$ .

Note that the  $s > 1$  case would have a strong experimental signal. It would have a very strong signal at  $t = \pi/2$  that is very short in time. Wineland's group [11] is independently trying to create such states.

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## Figure Captions

Figure 1. A three-dimensional plot of the even-coherent-state probability density,  $\rho_+$ , as a function of position,  $x$ , and time,  $t$ , for  $\alpha_0 = 2$ .

Figure 2. A three-dimensional plot of the odd-coherent-state probability density,  $\rho_-$ , as a function of position,  $x$ , and time,  $t$ , for  $\alpha_0 = 5^{1/2}$ .

Figure 3. The dashed curve is a plot of  $\psi_{Ess}(x)$  vs.  $x$ , with parameters  $\alpha = 2$  and  $q = 2$ . The normalization constant  $N_E = 0.08190$ . The solid curve is a plot of  $\psi_{s+}(x)$  vs.  $x$ , with parameters  $\alpha = 2$  (or  $x_0 = 2\sqrt{2}$ ,  $p_0 = 0$ ) and  $s = 3/2$ .

Figure 4. A three-dimensional plot of the even-squeezed-state probability density,  $\rho_{s+}$ , as a function of position,  $x$ , and time,  $t$ , for  $x_0 = 4$  and  $s = 2$ .

Figure 5. A three-dimensional plot of the odd-squeezed-state probability density,  $\rho_{s+}$ , as a function of position,  $x$ , and time,  $t$ , for  $x_0 = 4$  and  $s = 2$ .

Figure 6. A three-dimensional plot of the even-squeezed-state probability density,  $\rho_{s+}$ , as a function of position,  $x$ , and time,  $t$ , for  $x_0 = 4$  and  $s = 1/2$ .

Figure 7. A three-dimensional plot of the odd-squeezed-state probability density,  $\rho_{s+}$ , as a function of position,  $x$ , and time,  $t$ , for  $x_0 = 4$  and  $s = 1/2$ .